Effects of an almost resonant spatial thermal modulation in the Rayleigh-Benard problem: quasiperiodic behaviour

1987 J. Phys. A: Math. Gen. 20 1205

(http://iopscience.iop.org/0305-4470/20/5/029)

J. Phys. A: Math. Gen. 20 (1987) 1205-1214. Printed in the UK

Effects of an almost resonant spatial thermal modulation in the Rayleigh–Bénard problem: quasiperiodic behaviour

E M F Curado† and C Elphick

Laboratoire de Physique Théorique‡, Université de Nice, Parc Valrose, 06034 Nice Cedex, France

Received 1 July 1986

Abstract. We consider the effect of prescribed spatially periodic temperatures at the bounding walls of a 2D Boussinesq fluid on 1D pattern forming transitions. We determine the normal-form equation for the slowly varying pattern amplitude. Quasiperiodic behaviour is found when one takes into account the deviation of the external wavelength modulation from the critical value for the onset of classical Rayleigh-Bénard convection.

1. Introduction

In hydrodynamic convective systems if a control parameter—for example the Rayleigh number R in the classical Bénard problem—is greater than a critical value R_c, the system generally undergoes a transition from a stationary state (conduction state in the Bénard convection) to a spatially periodic pattern with a natural wavevector k_c. The selection of the preferred pattern wavevector k_c is a process not very well understood up to now. The natural framework to understand this selection mechanism is provided by convecting systems submitted to an external spatial modulation.

Recently some experiments have been carried out by Lowe *et al* (1983) and Lowe and Gollub (1985a, b) to measure the effects of applying a spatially periodic electric field with wavevector k_e to a layer of a nematic liquid crystal at the onset of an electrohydrodynamic instability. The competition between the natural and external periodicities leads to several interesting phenomena like commensurate states (i.e. stable states where the rolls are phase locked to the external forcing), incommensurate phases characterised by disturbances of the soliton type and chaotic phases.

From the theoretical point of view several works by Coullet *et al* (Coullet 1986, Coullet and Repaux 1986a, b) have studied these phenomena in a generic way, where the unforced system is supposed to be described by an evolution equation:

$$U_t = LU + N(U) \tag{1}$$

assumed to be invariant under spacetime translations ($T_1 \times T_1$ symmetry) and space reflections (Z_2 symmetry). In (1) U is a set of scalar fields describing a one-dimensional pattern forming transition and L, N denote linear and non-linear differential operators

† Supported in part by the Conselho Nacional de Desenvolvimento Cientifico et Tecnologico (CNPQ, Brazil). On leave of absence from Centro Brasileiro de Pesquisas Fisicas, CBPF, Rio de Janeiro, 22290 Brazil.
‡ Unité associée au CNRS.

depending on some control parameters, say R, etc. The imposed periodic spatial forcing breaks the translational invariance and the forced system becomes invariant under the discrete group of translations

$$x \to x + 2\pi n/k_e \qquad n \in \mathbb{Z}. \tag{2}$$

The corresponding normal form equation for the slowly spatio-temporal varying amplitude A of the critical mode has been derived by Coullet (1986) by noting that the broken SO(2) symmetry becomes the restricted symmetry

$$A \to A \exp(\mathrm{i}2\pi/n) \tag{3}$$

which leads in the almost resonant case $k_e = n(k_c + q)$ (q is termed the misfit) at the leading order in the amplitude equation to a SO(2) symmetry breaking term of the form

$$\alpha \bar{A}^{n-1} \tag{4}$$

where α is a small parameter measuring the forcing amplitude and $\bar{A}$ is the complex conjugate of A.

Nevertheless, we observe in the case $n = 1$ that, if the forced system admits a stationary solution finite at the bifurcation point $R = R_c$, then the external modulation will be coupled multiplicatively with U, implying that the forcing has no effect at the order given by (4). Therefore one is led to consider higher-order symmetry breaking terms in the amplitude equation. At the leading order they are of the form

$$\alpha^2 \bar{A} \qquad \alpha A^2 \qquad \alpha |A|^2. \tag{5}$$

Since the dominant non-linear term of the unforced normal form is cubic in A, $\bar{A}$, (5) clearly shows that A and α scale in the same way in the neighbourhood of the instability. This scaling behaviour has already been found by Kelley and Pal (1978a, b) in the quasi-conduction regime ($R \ll R_c$) but the stationary solution of this regime diverges when one approaches R_c and therefore one is forced to introduce a new scaling at R_c ($A \sim \alpha^{1/3}$).

The purpose of this paper is to study bidimensional convection with plane boundary walls that have some prescribed spatially periodic varying temperatures. We consider the almost resonant case $k_e = k_c + q$. The particular case studied here is remarkable since it admits a finite stationary solution even at $R = R_c$ and a global scaling behaviour $A \sim \alpha$. Also, quasiperiodic behaviour is found in some determined region of parameter space in accordance with the previous work by Coullet and Repaux (1986a).

2. The forced pattern and the amplitude equation

In suitable units bidimensional thermal convection for a Boussinesq fluid is described by the equations

$$T_t = \Delta T + J(\psi, T) \tag{6a}$$

$$\Delta\psi_t = \sigma(\Delta^2\psi - RT_x) + J(\psi, \Delta\psi) \tag{6b}$$

where ψ is the stream function (a pseudoscalar under Z_2), T represents the temperature, σ denotes the Prandtl number and $J(f, g) = f_x g_z - f_z g_x$. For ψ we employ stress-free boundaries at the top and bottom of the bounding walls

$$\psi = \psi_{zz} = 0 \qquad \text{on } z = 0, 1. \tag{7a}$$

For T the prescribed non-uniform boundary conditions are

$$T = 1 + \alpha \cos k_e x \qquad \text{on } z = 0 \tag{7b}$$

$$T = -\alpha \cos k_e x \qquad \text{on } z = 1 \tag{7c}$$

where α is a parameter measuring a sufficiently weak external forcing.

At the leading order in α the stationary solution of (6a) and (6b) is found to be

$$T_s = 1 - z + \alpha t(z) \cos k_e x + O(\alpha^2) \tag{8a}$$

$$\psi_s = \frac{\alpha}{k_e} \varphi(z) \sin k_e x + O(\alpha^2) \tag{8b}$$

where (see figure 1)

$$t(z) = -\frac{1}{3}\left(\frac{\sin \lambda(z-1) + \sin \lambda z}{\sin \lambda}\right) + \frac{2}{3(\cosh 2\rho - \cos 2\delta)}(\cos \delta z \cosh \rho(z-2)$$

$$- \cos \delta(z-2) \cosh \rho z + (z \to z+1)) \tag{9a}$$

$$\varphi(z) = t_{zz} - k_e^2 t \tag{9b}$$

and λ, δ, ρ are given by the relations

$$\lambda^2 = (Rk_e^2)^{1/3} - k_e^2 \tag{10a}$$

$$\rho^2 - \delta^2 = k_e^2 + \tfrac{1}{2}(Rk_e^2)^{1/3} \tag{10b}$$

$$2\rho\delta = \tfrac{1}{2}\sqrt{3}(Rk_e^2)^{1/3}. \tag{10c}$$

The stationary fluid velocities in the x and z directions are given by (figure 2)

$$v_x = (\alpha / k_e)\varphi_z \sin k_e x + O(\alpha^2) \tag{11a}$$

$$v_z = -\alpha\varphi \cos k_e x + O(\alpha^2). \tag{11b}$$

The corresponding pattern, called the forced pattern, is shown in figure 3 in the almost resonant case $k_e = k_c + q$; the figure clearly shows the effect of the misfit q.

It is worth remarking that the stationary solution (8a) and (8b) has no singularities when R, k_e approach the critical values for the classical problem with uniform heating $R_c = \frac{27}{4}\pi^4$, $k_c = \pi/\sqrt{2}$ (corresponding to a zero eigenvalue of the associated linear

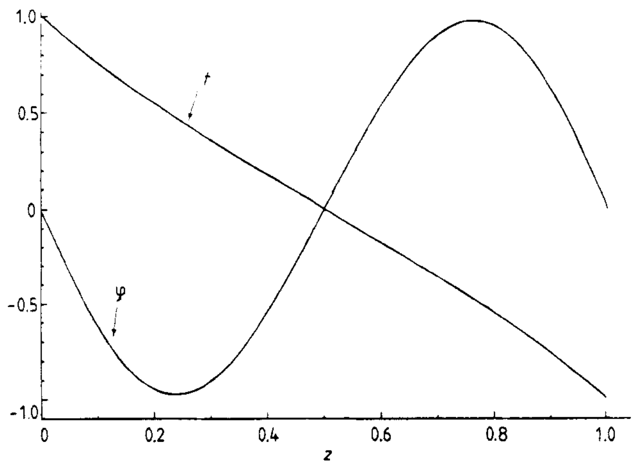

Figure 1. Graph of the function t, φ related to the forced pattern via (8a) and (8b). $q = 0.1$.

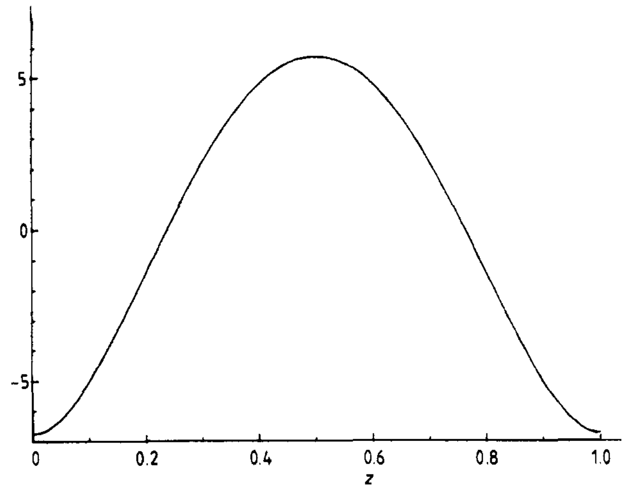

Figure 2. Graph of φ_z introduced in (11a). $q = 0.1$.

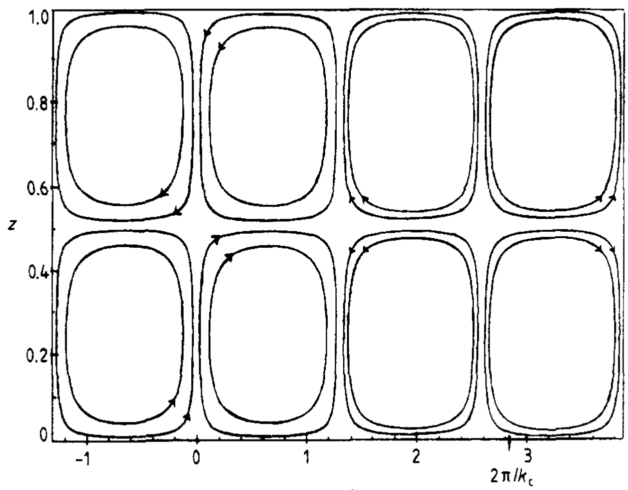

Figure 3. Roll structure of the almost resonant forced pattern for mercury ($\sigma = 0.025$) and misfit $q = 0.217$.

operator L in (6a) and (6b)), which in turn shows the validity of the analytic expansion in powers of α in the neighbourhood of these critical values. We note that when R, $k \to R_c$, k_c then $\lambda \to \pi$ and the first term in (9a) becomes $\frac{1}{3}\cos \pi z$.

In order to study the dynamics associated to (6a) and (6b) we write

$$U = \begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} T - T_s \\ \psi - \psi_s \end{pmatrix}$$

and perform an asymptotic expansion of U in terms of a slowly varying amplitude $A'(x, t)$ (varying in a scale much larger than $\max(2\pi/k_e, 2\pi/k_c)$):

$$U = A'\varphi_{k_c} \exp(ik_c x) \sin \pi z + \mathrm{cc} + \mathcal{U}(A', \bar{A}', \alpha, \mu; x, z) \tag{12}$$

where $\varphi_{k_c} \exp(ik_c x) \sin \pi z$ is the marginal mode of L, $\mu = \frac{3}{2}\pi^2[(R - R_c)/R_c]$ and $\mathcal{U}$

represents the centre manifold contribution to U. Classical methods developed by Coullet and Spiegel (1983) and Elphick *et al* (1986) allows one to reduce the equation satisfied by U to a normal-form equation for $A = A' \exp(-iqx)$

$$[(1+\sigma)/\sigma]A_t = (\mu - 4q^2 + \alpha^2 c_2)A + \alpha^2 c_1 \bar{A} - \tfrac{9}{8}\pi^4 |A|^2 A + 4(A_{xx} + 2iqA_x). \tag{13}$$

We note that equation (13) has a gradient structure

$$[(1+\sigma)/\sigma]A_t = -\delta\mathcal{F}/\delta\bar{A} \tag{14}$$

where the Lyapounov functional $\mathcal{F}$ is given by

$$\mathcal{F} = \int (-(\mu - 4q^2 + \alpha^2 c_2)|A|^2 - \tfrac{1}{2}\alpha^2 c_1 (A^2 + \bar{A}^2) + \tfrac{9}{16}\pi^4 |A|^4 + 4(|A_x|^2 + iq(A\bar{A}_x - \bar{A}A_x)))\,\mathrm{d}x. \tag{15}$$

From (14), one has

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = -2\int |A_t|^2\,\mathrm{d}x$$

and since $\mathcal{F}$ is bounded from below, equation (13) admits stationary solutions.

Let us say that the terms αA^2, $\alpha|A|^2$ have not been included in (13) since their contributions turn out to be of the form

$$\alpha q(c_3 A^2 + c_4 |A|^2). \tag{16}$$

Scaling analysis shows that (16) is a higher-order term and need not be considered in (13). We remark that, if a term like (16) is to be present in (13), then the gradient structure is broken (since $c_4 \neq 2c_3$).

The constants c_1 and c_2 in (13) are given by (the result of the integrals is omitted for the sake of brevity)

$$c_1 = I_2 - \tfrac{3}{8}\pi^2 I_1 \tag{17a}$$

$$c_2 = I_2 - I_3 \tag{17b}$$

where

$$I_1 = \int_0^1 t\varphi \cos 2\pi z\,\mathrm{d}z \tag{18a}$$

$$I_2 = \sum_{m\geq 1} \left(\int_0^1 (\varphi - \tfrac{3}{2}\pi^2 t)\sin \pi z \cos m\pi z\,\mathrm{d}z\right)\left(\int_0^1 \varphi \sin \pi z \cos m\pi z\,\mathrm{d}z\right) \tag{18b}$$

$$I_3 = \sum_{m\geq 1} (f_1(\sigma, m)J_{1m} - f_2(\sigma, m)J_{2m})J_{3m} \tag{18c}$$

$$\begin{Bmatrix} J_{1m} \\ J_{2m} \\ J_{3m} \end{Bmatrix} = \int_0^1 (2\pi \cos \pi z \sin m\pi z + m\pi \cos m\pi z \sin \pi z)\begin{Bmatrix} t \\ \varphi \\ \varphi + \tfrac{3}{2}\pi^2 t \end{Bmatrix} \tag{18d}$$

$$f_1(\sigma, m) = \frac{1}{2}\,\frac{[\gamma^{(1)} + \pi^2(m^2+2)][\gamma^{(2)} + \pi^2(m^2+2)](\gamma^{(1)} + \gamma^{(2)})}{\gamma^{(1)}\gamma^{(2)}(\gamma^{(2)} - \gamma^{(1)})} \tag{18e}$$

$$f_2(\sigma, m) = \frac{\gamma^{(1)}[\gamma^{(1)} + \pi^2(m^2+2)] + \gamma^{(2)}[\gamma^{(2)} + \pi^2(m^2+2)]}{\gamma^{(1)}\gamma^{(2)}(\gamma^{(2)} - \gamma^{(1)})} \tag{18f}$$

$$\gamma^{(1/2)} = \tfrac{1}{2}\pi^2(m^2+2)\left[-(1+\sigma) \pm \left((1-\sigma)^2 + \frac{54\sigma}{(m^2+2)^3}\right)^{1/2}\right]. \tag{18g}$$

Table 1. Numerical values of the normal-form coefficients c_1, c_2

σ	c_1	c_2
0.025	1.1877	6.6269
7.5	1.1877	8.2929
∞	1.1877	7.6174

Close formulae for the sums in (18*b*) and (18*c*) are difficult to find, but it is easy to show that they are convergent series yielding positive values for c_1 and c_2 for any $\sigma > 0$. Table 1 shows numerical values of c_1 and c_2 for mercury ($\sigma = 0.025$), water ($\sigma = 7.5$) and $\sigma = \infty$ (from (17*a*) c_1 does not depend on σ).

3. Transitions from the forced pattern to locked patterns

A locked pattern is a stationary homogeneous solution $A = Q\,\mathrm{e}^{\mathrm{i}\theta}$ of (13). One readily obtains the equations

$$-\alpha^2 c_1 \sin 2\theta = 0 \tag{19a}$$

$$(\mu - 4q^2 + \alpha^2 c_2)Q - \tfrac{9}{8}\pi^4 Q^3 + \alpha^2 c_1 \cos 2\theta = 0. \tag{19b}$$

The solutions $\theta = \pi/2, 3\pi/2$ of (19*a*) are found to be unstable under phase perturbations. The bifurcation diagram associated to (19*b*) is shown in figure 4 (a broken curve represents the locked pattern with $\theta = \pi/2, 3\pi/2$). The transition from the forced pattern $A = 0$ to a locked pattern, stable under homogeneous perturbation, occurs on the critical surface in parameter space (μ, q, α):

$$\mu_c = 4q^2 - \alpha^2(c_1 + c_2) \tag{20}$$

which measures the shift in the critical Rayleigh number R_c due to the forcing.

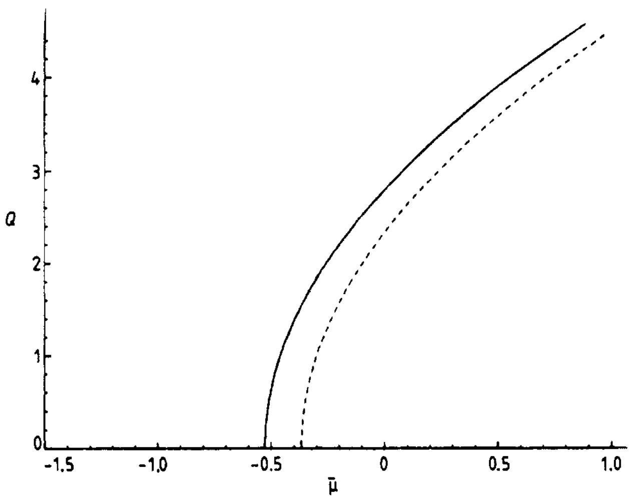

Figure 4. Bifurcation diagram associated with (19*b*). A broken curve represents the unstable locked pattern $\theta = \pi/2, 3\pi/2$. The amplitude is plotted as a function of $\bar{\mu} = \mu - 4q^2$ and for $\sigma = 0.025$.

4. Stability of the forced pattern and quasiperiodic behaviour

The forced pattern ($A=0$) is stable under homogeneous perturbations for $\mu<\mu_c$ and any value of q. The stability under inhomogeneous perturbations of wavevector k is studied by linearising (13) and imposing the existence of a marginal mode. This leads to the following marginal surface:

$$(\mu-4q^2-k^2)^2=16k^2q^2+\alpha^4c_1^2. \tag{21}$$

The minimum of (21) is found to be at

$$\mu_c'=-\alpha^2(c_2+\alpha^2c_1/16q^2) \tag{22}$$

and the optimal wavevector being

$$k_1=\frac{(64q^4-\alpha^4c_1^2)^{1/2}}{8q}. \tag{23}$$

Therefore for $\mu>\mu_c'$ and $8q^2>\alpha^2c_1$ the forced pattern is unstable under finite wavelength perturbations. The marginal mode is readily found to be

$$\begin{aligned}A&=[(8q^2-\alpha^2c_1)^{1/2}-(8q^2+\alpha^2c_1)^{1/2}\exp(\mathrm{i}k_1x)\\&\qquad+[(8q^2-\alpha^2c_1)^{1/2}+(8q^2+\alpha^2c_1)^{1/2}]\exp(-\mathrm{i}k_1x)\\&=a_1\exp(\mathrm{i}k_1x)+a_2\exp(-\mathrm{i}k_1x)\end{aligned} \tag{24}$$

and therefore this instability marks the appearance of quasiperiodic behaviour since the actual pattern is characterised by three wavevectors k_c+q, $k_c\pm k_1+q$.

The next step is to study the stability of the quasiperiodic pattern. To do so one has to consider the non-linearities in (13), neglected in the linear analysis, introduce a new slowly varying amplitude B and perform an asymptotic expansion:

$$A=a_1B\exp(\mathrm{i}k_1x)+a_2\bar{B}\exp(-\mathrm{i}k_1x)+\mathscr{A}(B,\bar{B},\mu-\mu_c';x)$$

where $\mathscr{A}$ is the centre manifold contribution to A.

The equation for B is easily obtained:

$$\left(\frac{1+\sigma}{\sigma}\right)B_t=(\mu-\mu_c')B-\frac{9\pi^4}{32q^2}(64q^4+\alpha^4c_1^2)|B|^2B+\frac{k_1^2}{4q^2}B_{xx} \tag{25}$$

which tell us the supercritical nature and stability of the quasiperiodic pattern.

Since $\mu_c'<\mu_c$ the transition to quasiperiodic behaviour occurs before the transition to locked patterns in the parameter region $q^2>\frac{1}{8}\alpha^2c_1$. We finally mention that a similar analysis shows that the quasiperiodic behaviour arising from the locked patterns has a subcritical nature and therefore implies its instability.

5. The actual quasiperiodic pattern

At the leading order in A and α the actual quasiperiodic pattern in the neighbourhood of μ_c' ($\mu-\mu_c'>0$) is given by

$$\begin{pmatrix}T\\\psi\end{pmatrix}=\begin{pmatrix}T_s\\\psi_s\end{pmatrix}+\sin\pi z\left[\begin{pmatrix}1\\\mathrm{i}3k_c\end{pmatrix}A\exp[\mathrm{i}(k_c+q)x]+\mathrm{cc}\right]+\mathrm{O}(\alpha^2,A^2,\bar{A}^2,|A|^2,\alpha A,\alpha\bar{A}) \tag{26}$$

where A is given by

$$A=\eta[a_1\exp(\mathrm{i}k_1x)+a_2\exp(-\mathrm{i}k_1x)] \tag{27}$$

$$\eta=\left(\frac{8q^2(\mu-\mu_c')}{64q^4+\alpha^4c_1^2}\right)^{1/2}. \tag{28}$$

Explicitly the quasiperiodic fluid temperature is given by

$$T(x,z)=1-z+\alpha t(z)\cos k_ex+2\eta\sin\pi z[a_1\cos(k_c+k_1+q)x+a_2\cos(k_c-k_1+q)] \tag{29}$$

and the quasiperiodic fluid velocities in the x and z direction are given by

$$v_x(x,z)=\frac{\alpha}{k_e}\varphi_z\sin k_ex-6\pi k_c\eta\cos\pi z[a_1\sin(k_c+k_1+q)x+a_2\sin(k_c-k_1+q)x] \tag{30a}$$

$$v_2(x,z)=-a\varphi\cos k_ex+6k_c\eta\sin\pi z[(k_c+k_1+q)a_1\cos(k_c+k_1+q)x + (k_e-k_1+q)a_2\cos(k_c-k_1+q)x]. \tag{30b}$$

Figure 5 shows the field lines associated with this quasiperiodic pattern.

The quasiperiodic behaviour is better analysed by defining the following map:

$$x_n\to P(x_n)=\left(\frac{l+1}{l}\right)x_n-(n-\tfrac{1}{4}) \tag{31}$$

where x_n denotes the position of the nth roll pair (middle point of the second roll) and l is the wavelength of the external forcing. With the help of (31) one defines a phase variable (3):

$$\phi_n=2\pi(P(x_n)-x_n) \tag{32}$$

which measures the deviation in position of the nth roll pair from the locked value nl (fixed point of P). We can interpret the map P as a spatial Poincaré section for the

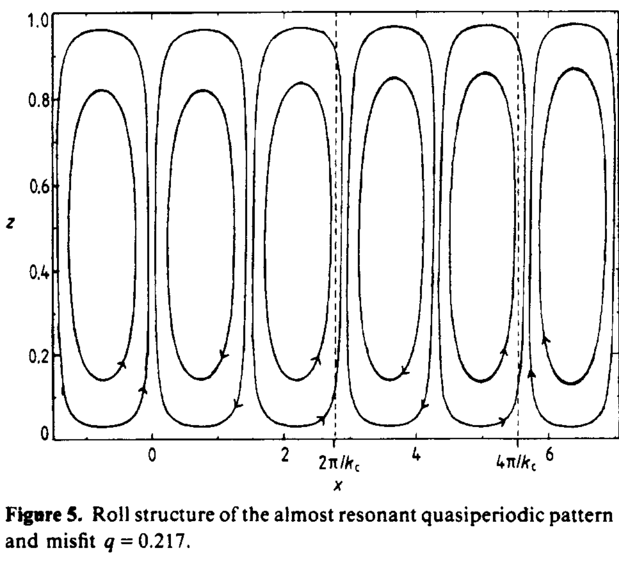

Figure 5. Roll structure of the almost resonant quasiperiodic pattern for mercury ($\sigma=0.025$) and misfit $q=0.217$.

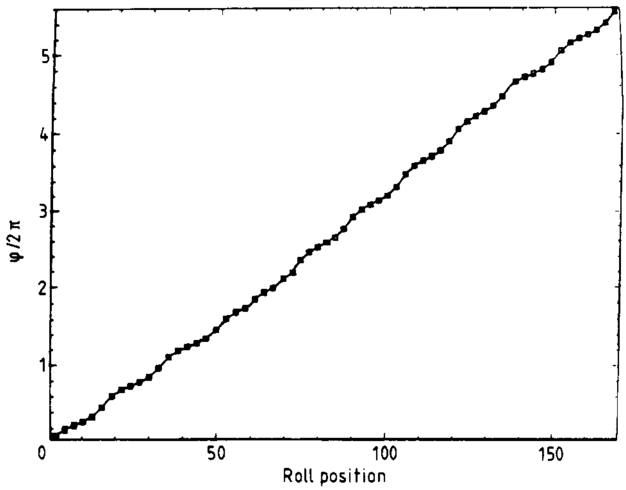

Figure 6. Roll phase as a function of the roll-pair position ($\sigma = 0.025$, $q = 0.217$).

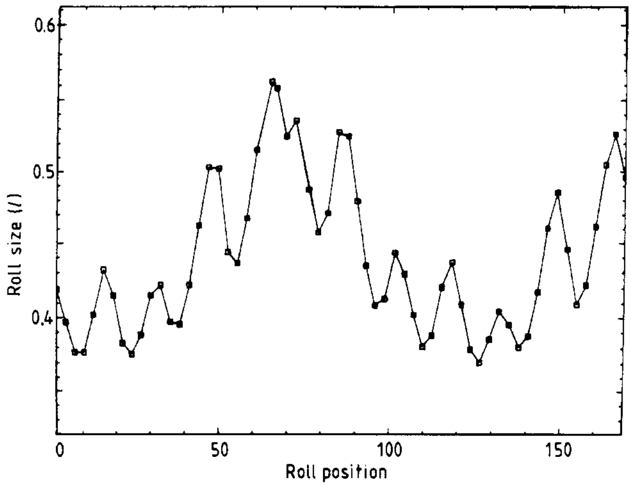

Figure 7. Roll size in l units as a function of the roll-pair position.

continuous roll-phase variable. Figure 6 shows the behaviour of the phase variable for 60 roll pairs. We also show in figure 7 the change in roll size as a function of the roll position.

6. Conclusions

We have studied in this work, following the ideas of the recent work by Coullet *et al* (Coullet 1986, Coullet and Repaux 1986a, b), how quasiperiodicity arises in a Rayleigh–Bénard system subjected to a spatially periodic thermal forcing. This behaviour is found when one takes into account the misfit between the external wavevector and the critical wavevector for the onset of Rayleigh–Bénard convection. In the present case the corresponding amplitude equation contains higher-order SO(2) symmetry breaking terms than the ones considered by Coullet (1986), which are of the form $\alpha\bar{A}^{n-1}$ ($k_e = n(k_c + q)$). If $n = 1$ (the problem studied here) this yields a constant term in the normal form. Since U (12) is multiplicatively coupled with the forcing the constant term cannot be present and one has to consider higher-order terms in the normal form. We have found that the leading one is proportional to $\alpha^2\bar{A}$.

For the cases $n \neq 1$, it is not difficult to compute the coefficients of the terms predicted by Coullet (1986): they all vanish due to Boussinesq symmetry, and again one has to go to higher orders. For instance, in the case $n = 2$, a symmetry breaking term $\alpha\bar{A}_x$ is present in the normal form. Finally let us say that for forced convection in rotation terms of the form $\alpha\bar{A}^{n-1}$ are likely to be present in the amplitude equation. Work related to the last ideas is in progress.

Acknowledgments

We are grateful to P Coullet for having introduced us to this topic and for stimulating discussions. We thank L Gil, J Lega, H Politano and D Repaux for fruitful discussions. Also we acknowledge interesting suggestions from G Iooss and H Haken. One of us (EMFC) thanks the CNPq (Brazil) for financial support.

References

Coullet P 1986 *Phys. Rev. Lett.* **56** 724-7
Coullet P and Repaux D 1986a *J. Am. Math. Soc.* to appear
—— 1986b *Europhys. Lett.* to appear
Coullet P and Spiegel E A 1983 *SIAM J. Appl. Math.* **43** 776 -821
Elphick C, Tirapegui E, Brachet M E, Coullet P and Iooss G 1986 *Physica* D to appear
Kelley R E and Pal D 1978a *J. Fluid Mech.* **86** 433-56
—— 1978b *Proc. 6th Int. Heat Transfer Conf. Toronto*
Lowe M and Gollub J 1985a *Phys. Rev. Lett.* **55** 2575-8
—— 1985b *Phys. Rev.* A **31** 3893-7
Lowe M, Gollub J and Lubensky T C 1983 *Phys. Rev. Lett.* **51** 786-9